1. Compute the following derivatives. (Simplify your answers when possible.)

   (a) $f'(x)$ where $f(x) = \dfrac{x}{1-x^2}$

   (b) $f'(x)$ where $f(x) = \ln(\cos x) - \dfrac{1}{2}\sin^2(x)$

   (c) $f^{(5)}(x)$, the fifth derivative of f, where $f(x) = xe^x$

2. Find the equation of the tangent line to the "astroid" curve defined implicitly by the equation

$$x^{2/3} + y^{2/3} = 4$$

at the point $(-\sqrt{27}, 1)$.

3. A particle is moving along a vertical axis so that its position y (in meters) at time t (in seconds) is given by the equation

$$y(t) = t^3 - 3t + 3, \quad t \geq 0.$$

Determine the total distance traveled by the particle in the first three seconds.

4. State the product rule for the derivative of a pair of differentiable functions f and g using your favorite notation. Then use the DEFINITION of the derivative to prove the product rule. Briefly justify your reasoning at each step.

5. Does there exist a set of real numbers a, b and c for which the function

$$f(x) = \begin{cases} \tan^{-1}(x) & x \leq 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \geq 2 \end{cases}$$

is differentiable (i.e. everywhere differentiable)? Explain why or why not. (Here $\tan^{-1}(x)$ denotes the inverse of the tangent function.)

6. Suppose that f satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose further that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$

(a) Find $f(0)$.

(b) Find $f'(0)$.

(c) Find $f'(x)$.

First attempt: 29/7/25

2 hrs 45 mins

1.

8.5/10

(a) $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{1\cdot(1-x^2) - x\cdot(-2x)}{(1-x^2)^2}$$

$$= \frac{-x^2+2x^2+1}{(1-x^2)^2}$$

$$= \frac{x^2+1}{(x^2-1)^2}$$

(b) $f(x) = \ln(\cos x) - \frac{1}{2}\sin^2(x)$

$$f'(x) = \frac{1}{\cos x}\cdot(-\sin x) - \frac{1}{2}\cdot 2\sin x\cdot\cos x$$

$$= -\frac{\sin x}{\cos x} - \sin x\cos x$$

$$= -\tan x - \sin x\cos x$$

Model answer: $-\sin x\left(\dfrac{1+\cos^2 x}{\cos x}\right)$

(c) $f(x) = xe^x$

Let $u = x$ and $v = e^x$.

$\Rightarrow f(x) = uv$

$$f^{(5)}(x) = u^{(5)}v + \binom{5}{1}u^{(4)}v^{(1)} + \dots + uv^{(5)}$$

All derivatives of $u = 0$ except u'.

$$\therefore f^{(5)}(x) = \binom{5}{4}u^{(1)}v^{(4)} + uv^{(5)}$$

$$= 5(1)e^{x} + xe^{x}$$

$$= e^{x}(x+5)$$

2.

$$x^{2/3} + y^{2/3} = 4$$

$$y^{2/3} = 4 - x^{2/3}$$

$$y^{1/3} = \sqrt{4 - x^{2/3}}$$

$$\frac{d}{dx}\left(x^{2/3} + y^{2/3}\right) = \frac{d}{dx}4$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

$$= -\frac{\sqrt{4 - x^{4/3}}}{x^{1/3}}$$

$$y = mx + c$$

$$\Rightarrow \quad y = -y^{1/3} x^{2/3} + C$$

At $(-\sqrt{27}, 1)$,

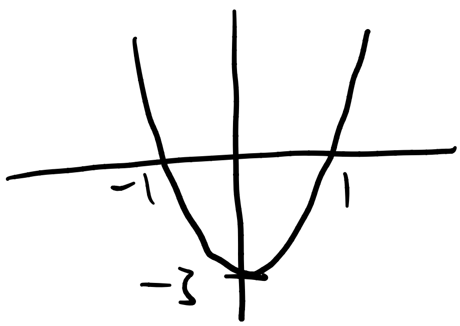
$$C = 1 + 1\left(-\sqrt{27}\right)^{2/3}$$
$$= 1 + 3$$
$$= 4$$

$\therefore$ Tangent line: $y = -\sqrt{4 - x^{2/3}}\, x^{2/3} + 4$

3.

$$y(t) = t^3 - 3t + 3, \quad t \geq 0$$

$$\frac{dy}{dt} = 3t^2 - 3$$



From $t=0$ to $t=1$, $y(t)$ is decreasing. $\Delta y = y(1) - y(0) = -2$

From $t=1$ to $t=3$, $y(t)$ is increasing. $\Delta y = y(3) - y(1)$
$$= 21 - 1$$
$$= 20$$

$\therefore$ total distance travelled $= 22$

4. $h(x) = f(x) \cdot g(x)$

Product rule: $h'(x) = f'(x)g(x) + f(x)g'(x)$

Proof:

$$h'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x)}{\Delta x}$$

$$- \frac{f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{(f(x+\Delta x) - f(x))g(x+\Delta x)}{\Delta x} + \frac{(g(x+\Delta x) - g(x))f(x)}{\Delta x} \right]$$

$$= f'(x)g(x) + g'(x)f(x) \qquad 1/2$$

$$\therefore h'(x) = f'(x)g(x) + f(x)g'(x)$$

5.

$$f(x) = \begin{cases} \tan^{-1}(x) & x \le 0 \\ ax^2 + bx + c, & 0 < x < 2 \\ x^3 - \frac{1}{4}x^2 + 5, & x \ge 2 \end{cases}$$

$\tan^{-1}(x)$, ax^2+bx+c and $x^3 - \frac{1}{4}x^2+5$ are all differentiable functions.

At $x=0$, $\tan^{-1}(x) = 0$ and $ax^2+bx+c = c$

At $x=2$, $ax^2+bx+c = 4a+2b+c$ and $x^3 - \frac{1}{4}x^2+5 = 12$

$\Rightarrow c = 0$ $\qquad \Rightarrow 2a+b=6$

$4a+2b+c=12$

At $x=0$, $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$

$$\frac{1}{1+(0)^2} = 2a(0)+b$$

$$\Rightarrow \quad b = 1$$

At $x=2$, $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x)$

$$2a(2)+b = 3(2)^2 - \frac{1}{2}(2)$$

$$4a+b = 12-1$$

$$4a+b = 11$$

With $b=1$, $4a+b=11 \Rightarrow a = \frac{5}{2}$

$2a+b = 6 \Rightarrow a = \frac{5}{2}$

$\therefore$ $f(x)$ is differentiable as $f(x)$ ~~is~~ continuous everywhere and its limit exists at all points with $a=\frac{5}{2}$, $b=1$, $c=0$.

6. $f(x+y) = f(x)+f(y)+x^2y+xy^2$

$$\lim_{x\to 0} \frac{f(x)}{x} = 1$$

(a) $f(0)$

$\Rightarrow x+y=0$

$y=-x$

$\Rightarrow f(0)= f(x)+f(-x) -x^3+x^3$

$= f(x)+f(y)$

If $y=0$, then $f(x)= f(x)+f(0)$.

$\therefore f(0)=0$

(b) $f'(0)$

$$\lim_{x\to 0} \frac{f(x)}{x}$$

$$= \lim_{x\to 0} \frac{f(0+x)-f(0)+f(0)}{x}$$

$$= f'(0) + \lim_{x\to 0} \frac{f(0)}{x}$$

$$= f'(0) + \lim_{x\to 0} \frac{f(x)+f(y)}{x}$$

$$= f'(0) + 1 + \lim_{x\to 0} \frac{f(y)}{x}$$

$$\Rightarrow 1 = f'(0) + 1 + \lim_{x\to 0} \frac{f(y)}{x}$$

$$\Rightarrow f'(0) = -\lim_{x\to 0} \frac{f(y)}{x}$$

$\because f(y) = -f(x)$

$$\therefore f'(0) = \lim_{x\to 0} \frac{-(-f(x))}{x} = \lim_{x\to 0} \frac{f(x)}{x} = 1$$

(c) $f'(x)$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Let $\Delta x = y$.

$$\Rightarrow \lim_{y \to 0} \frac{f(x+y) - f(x)}{y}$$

$$= \lim_{y \to 0} \frac{f(x)+f(y)+x^2y+xy^2 - f(x)}{y}$$

$$= \lim_{y \to 0} \frac{f(y) + y(x^2+xy)}{y}$$

$$= \lim_{y \to 0} \frac{f(y)}{y} + \lim_{y \to 0} (x^2+xy)$$

$$= 1 + x^2$$

$\therefore f'(x) = x^2 + 1$.